The influence of radiative transfer on cellular convection

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SUMMARY

This paper presents an approximate solution of the problem of the onset of convection between plane-parallel plates heated from below when the fluid between them absorbs and emits thermal radiation. A complete solution to this problem would be extremely difficult, and the equation of radiative transfer is therefore developed in two approximate forms, one appropriate to an opaque medium, the other to a transparent medium. This equation is then combined with the dynamical equations of the problem.

The initial static state is investigated by use of the Milne-Eddington approximation, and it is shown that there can be very large variations of temperature near to the boundaries.

The conditions for marginal stability are investigated both for motions which are restricted to the temperature boundary layer, and for motions which take place in the body of the fluid. In the former case it is found that a complete solution is provided by the approximate form for a transparent medium, and in the latter case a reasonable interpolation has to be made between results for the two approximate forms in order to complete the solution.

The effect of radiative transfer both on the initial static state and on the dynamical equations is such that the fluid is stabilized. This stabilization could probably be detected in the laboratory under certain conditions. In the earth's atmosphere the critical Rayleigh number for large scale motions may be increased by a factor 10^5, while at the surface of the solar photosphere the factor may be as large as 10^{12}.

1. INTRODUCTION

The gaseous envelope of a star or planet must pass out to space a flux of radiative heat. This usually gives rise to motions in the atmosphere, so that the atmosphere will not be in radiative equilibrium but will have sources and sinks of radiative energy distributed through it.

Investigators of the earth's atmosphere always compute the field of radiative heating from observed temperature profiles, and attempt to find the field of motion consistent with it. This does not go to the heart of the problem, however, for we wish to know what fields of temperature and motion are demanded by the radiative boundary conditions, not only whether they are consistent with one another.

The aim of this paper is to explore the possibility of formulating the equations of motion and radiative transfer in a way which allows them to be solved directly in accordance with the boundary conditions. The problem chosen is that of the marginal stability of a fluid between parallel plates, heated from below. This is probably the simplest problem of its type, and the same problem in the absence of radiative transfer has been treated by many writers (see, for example, Pellew & Southwell (1940)). The fluid is assumed to be homogeneous in composition and incompressible, and the temperature to vary only slightly. The absorption coefficient of the fluid is assumed to be the same at all wavelengths, and to be independent of the physical state. The upper and lower boundaries to the fluid are assumed to be black bodies.

2. The dynamical equations

These were treated very fully by Pellew & Southwell (1940), whom we shall follow closely. The following symbols will be used:

ν = kinematic viscosity,
α = coefficient of expansion,
g = gravitational acceleration,
x, y, z = rectangular coordinates (z vertical and perpendicular to the plates),
w = velocity component in the z direction,
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla_1^2 + \frac{\partial^2}{\partial z^2}$,
θ = temperature,
θ_0 = temperature in the initial static state, $\beta = \frac{\partial \theta_0}{\theta z}$,
$\theta' = \theta - \theta_0$,
H = rate of radiative heating per unit volume of fluid,
H_0 = the same for the initial static state,
$H' = H - H_0$,
k = molecular thermal diffusivity,
s = heat content of the fluid per cm^3 per degree.

The linearized equation governing the heat transfer by fluid motion in a steady state was given by Pellew & Southwell in the form

$$-\nu\nabla^4 w = \alpha g \nabla_1^2 \theta'. \tag{1}$$

Their derivation of equation (1) made use of the fact that, in the absence of radiative transfer, there is constant temperature gradient through the fluid in the initial static state. The same equation can, however, be obtained with no extra difficulty if the temperature gradient varies with z.

The condition for a steady distribution of temperature is

$$w\beta = H/s + k\nabla^2\theta, \tag{2}$$

which relates the convective, radiative and diffusive heating; and the initial static state is determined by

$$0 = H_0/s + k\nabla^2\theta_0. \tag{3}$$

Temperature will be assumed constant over the upper and lower boundaries, and therefore H_0 and θ_0 are functions of z only. The vector flux of radiative energy in the initial static state will be in the z direction, and will also be a function of z only. If F_z is the z component of this flux, then $H_0 = -dF_z/dz$, and we may write (3) in the integrated form

$$F_z - ks\beta = \text{const.} \tag{4}$$

Combining (2) and (3), we find

$$w\beta = H'/s + k\nabla^2\theta'. \tag{5}$$

Pellew & Southwell assume that w and θ' are separable functions of x, y and z, and that

$$\nabla_1^2 w = -\frac{a^2}{h^2}w, \tag{6}$$

where h is the distance between the plates and a is a 'characteristic number'; the same assumptions will be adopted here.

Further let

$$\zeta = \left(\frac{z}{h} - \tfrac{1}{2}\right), \quad \text{and} \quad \frac{\partial^2}{\partial\zeta^2} = D^2,$$

so that

$$\nabla^2 w = \frac{1}{h^2}(D^2 - a^2)w. \tag{7}$$

The elimination of θ' between equations (1) and (5) leads to

$$\nabla_1^2 w\beta = \frac{\nabla_1^2 H'}{s} - \frac{k\nu}{\alpha g}\nabla^6 w. \tag{8}$$

Since β is a function of z only, (8) can be rewritten, with the use of (6) and (7), as

$$\frac{a^2}{h^2}w\beta = -\frac{\nabla_1^2 H'}{s} + \frac{k\nu}{\alpha g}\frac{(D^2-a^2)^3 w}{h^6}. \tag{9}$$

We shall seek values of the Rayleigh number

$$R = -\frac{\bar{\beta}\alpha g h^4}{k\nu}, \tag{10}$$

where $\bar{\beta}$ is the mean value of β throughout the fluid, at which there is marginal stability. In general R will be a function of a^2, and the condition for marginal stability is found by minimizing R with respect to this parameter, together with any other variable parameters which are introduced.

The simplest boundary conditions will be used; these are that the plates are free and conducting surfaces, i.e.

$$w = D^2 w = \theta = 0 \quad \text{at} \quad \zeta = \pm\tfrac{1}{2} \tag{11a}$$

Since temperature is assumed constant over the boundaries, these conditions also require that

$$D^4 w = 0 \quad \text{at} \quad \zeta = \pm\tfrac{1}{2}. \tag{11 b}$$

With these boundary conditions the problem without radiative transfer (which will be referred to as the conventional Rayleigh problem) can be solved exactly, giving the critical Rayleigh number as

$$R_c = \frac{27\pi^4}{4} = 657. \tag{12}$$

3. Approximate forms for $\nabla_1^2 H'$

The function $\nabla_1^2 H'$ can be expressed in the form of a complicated integral over the whole fluid. The solution of even a very simple equation of the type $H_0 = 0$ is a matter of great difficulty (see, for example, Chandrasekhar 1950), and can only be achieved by iterative methods. In view of the present state of mathematical techniques, an exact solution of (9) does not seem possible. There are, however, two simple approximate forms for $\nabla_1^2 H'$, one valid when the fluid is optically thick and the other when it is optically thin. When numerical values are inserted, there is remarkably little doubt about the way in which these two solutions join up.

The equation of transfer for the problem is (Kourganoff 1953)

$$\frac{dI(\mathbf{s})}{ds} = \kappa[B - I(\mathbf{s})], \tag{13}$$

where κ is the coefficient of absorption per unit volume, $I(\mathbf{s})$ the intensity of radiation in the direction of the vector, $\mathbf{s}$, B the Planck black-body intensity, and ds is an infinitesimal displacement in the $\mathbf{s}$ direction. The radiative heating rate is

$$H = -\int \frac{dI(\mathbf{s})}{ds}\, d\omega, \tag{14}$$

where ω is an element of solid angle, and the integral is taken over a solid angle of 4π. Since the black-body intensity is isotropic, we have from (13) and (14)

$$H = -4\pi\kappa B + \kappa \int I(\mathbf{s})\, d\omega. \tag{15}$$

The first term on the right hand side of (15) represents the cooling at a point in the fluid due to the emission of thermal radiation at the local temperature. The second term represents the heating due to absorption of radiation emitted by other elements of the fluid and by the boundaries. The mean free path of the radiation is κ^{-1}, and the main contribution to this second term will come from points spaced at about this distance from the point under consideration. From (6) the dimensions of a convection cell are seen to be of the order of h/a; so, if $\kappa^{-1} \gg h/a$, the irradiation at each point will originate either far outside the cell which contains the point, or from the boundaries. In this case the irradiation will not vary much over distances comparable with a cell dimension, and the variability of H will be chiefly due to the variability of B. Hence

$$\nabla_1^2 H \doteqdot -4\pi\kappa\nabla_1^2 B. \tag{16}$$

This will be called the 'transparent' approximation to equation (15), valid when $\kappa^2 h^2 \ll a^2$.

By Stefan's law we have that

$$\pi B = \sigma\theta^4, \tag{17}$$

where σ is Stefan's constant. For a small temperature range we may write approximately

$$\frac{\partial B}{\partial x} = \frac{4\sigma}{\pi}\theta^3\frac{\partial\theta}{\partial x} = \underset{\sim}{Q}\frac{\partial\theta}{\partial x}, \tag{18}$$

and treat $\underset{\sim}{Q}$ as a constant. Since θ_0 and H_0 are not functions of x and y, we have from (18), (16) and (1) that

$$\nabla_1^2 H' \doteqdot \frac{4\pi\underset{\sim}{Q}\kappa\nu}{\alpha g}\nabla^4 w. \tag{19}$$

For large κ, H can be expanded in a power series in terms of κ^{-2}. A formal solution of equation (13) is

$$I(\mathbf{s}) = e^{-\kappa s}\int_q^s \kappa e^{\kappa\sigma}B(\sigma)\,d\sigma, \tag{20}$$

where q remains to be determined from the boundary conditions. The boundary conditions will have an appreciable effect only at distances less than κ^{-1} from the boundary, and outside these regions we may neglect the contribution from the lower limit of the integral in (20). Successive partial integrations then yield

$$I(\mathbf{s}) = B - \kappa^{-1}\frac{dB}{ds} + \kappa^{-2}\frac{d^2B}{ds^2} - \kappa^{-3}\frac{d^3B}{ds^3}\ \ldots\ ; \tag{21}$$

and hence using (14), we obtain

$$H = -\int\frac{dB}{ds}\,d\omega + \kappa^{-1}\int\frac{d^2B}{ds^2}\,d\omega - \kappa^{-2}\int\frac{d^3B}{ds^3}\,d\omega + \ldots \tag{22}$$

Now

$$\frac{dB}{ds} = \mu_1\frac{\partial B}{\partial x} + \mu_2\frac{\partial B}{\partial y} + \mu_3\frac{\partial B}{\partial z}, \tag{23}$$

where μ_1, μ_2 and μ_3 are the three directional cosines of $\mathbf{s}$. B does not depend on the direction of $\mathbf{s}$, nor do its partial differentials with respect to x, y, z. A typical term in the first integral in (22) is therefore $\partial B/\partial x\ \int\mu_1\,d\omega$, which is zero. The second integral involves integrals like $\int\mu_1\mu_2\,dw$, which are zero, and integrals like $\int\mu_1^2\,d\omega$, which equal $\frac{4}{3}\pi$. Hence

$$\int\frac{d^2B}{ds^2}\,d\omega = \frac{4\pi}{3}\nabla^2 B. \tag{24}$$

Similar arguments show that the third term on the right hand side of (22) is zero, and that the fourth is smaller than the second by a factor of the order of $a^2/(\kappa^2h^2)$. Thus, provided $\kappa^2h^2 \gg a^2$,

$$\nabla_1^2 H' \doteqdot -\frac{4\pi Q\nu}{3\kappa\alpha g}\nabla^6 w. \tag{25}$$

We will refer to (25) as the 'opaque' approximation. This is essentially the equation found by Brunt (1944) in his attempt to form an analogy between radiative and conductive heat transfer. The quantity $4\pi Q/(3\kappa)$ has the dimensions of conductivity, and so we introduce the non-dimensional ratio

$$\chi = \frac{4\pi Q}{3\kappa k s}. \tag{26}$$

On substituting (18) and (25) into (9), we find that, for $\kappa^2h^2 \gg a^2$,

$$Rw\frac{\beta}{\bar{\beta}} = -\frac{(D^2-a^3)^3}{a^2}(1+\chi)w, \tag{27}$$

and, for $\kappa^2h^2 \gg a^2$,

$$Rw\frac{\beta}{\bar{\beta}} = -\frac{(D^2-a^2)^2}{a^2}\left\{(D^2-a^2)-3\kappa^2h^2\chi\right\}w. \tag{28}$$

It is of interest to note that so far no use has been made of the radiative boundary conditions in deriving (27) and (28), and that these equations are equally valid if the boundaries are black bodies or mirrors. The nature of the boundaries affects only β, the initial temperature gradient.

4. The initial static state

In the initial static state all quantities will be functions of z only, and the equation of transfer (13) becomes

$$\mu_3\frac{dI}{dz} = \kappa[B-I]. \tag{29}$$

Iterative solutions of one-dimensional radiative equilibrium problems all show that remarkably accurate results can be obtained by assuming a suitable simple form for the angular distribution of intensity. One of the simplest assumptions is the Milne–Eddington approximation:

$$\begin{aligned} I(\mu_3, z) &= I_+(z), \quad \text{for} \quad 0 < \mu_3 \leqslant 1, \\ I(\mu_3, z) &= I_-(z), \quad \text{for} \quad -1 \leqslant \mu_3 < 0. \end{aligned} \tag{30}$$

By use of this approximation a simple differential equation can be formulated for the radiative flux:

$$F_z = \int \mu_3 I \, d\omega. \tag{31}$$

On integrating (29) over a solid angle of 4π, we find

$$\frac{dF_z}{dz} = 4\pi\kappa B - 2\pi\kappa(I_+ + I_-), \tag{32}$$

and, if (29) is first multiplied by μ_3,

$$\frac{2\pi}{3}\frac{d}{dz}[I_+ + I_-] = -\kappa F_z. \tag{33}$$

Elimination of $(I_+ + I_-)$ from (32) and (33) gives

$$\frac{d^2F_z}{dz^2} = 4\pi\kappa\frac{dB}{dz} + 3\kappa^2 F_z. \tag{34}$$

Elimination of B between (4), (18) and (34) gives the differential equation of the problem, and we now require the boundary conditions at the upper boundary ($z = h$) the molecular diffusion will ensure continuity of temperature, and therefore the downward intensity I_- will be equal to $B(h)$. Substituting $I_- = B(h)$ in (32), we get

$$\left(\frac{dF_z}{dz}\right)_h = 2\pi\kappa[I_- - I_+]. \tag{35}$$

But, from (31),

$$F_z(h) = \pi[I_+ - I_-], \tag{36}$$

and therefore

$$\left(\frac{dF_z}{dz}\right)_h = -2\kappa F_z(h). \tag{37}$$

Similarly, at the lower boundary,

$$\left(\frac{dF_z}{dz}\right)_0 = 2\kappa F_z(0). \tag{38}$$

The solution of the problem is now straightforward, and we find

$$\left.\begin{aligned} \frac{\beta}{\bar{\beta}} &= L\cosh\lambda\zeta + M, \\ L &= \chi\left[\frac{2\chi}{\lambda}\sinh\tfrac{1}{2}\lambda + (3+3\chi)^{1/2}\sinh\tfrac{1}{2}\lambda + \cosh\tfrac{1}{2}\lambda\right]^{-1}, \\ M &= \frac{L}{\chi}[(3+3\chi)^{1/2}\sinh\tfrac{1}{2}\lambda + \cosh\tfrac{1}{2}\lambda], \end{aligned}\right\} \tag{39}$$

where

$$\lambda^2 = 3\kappa^2h^2(1+\chi). \tag{40}$$

According to (39), $\beta/\bar{\beta} \to 1$ if either λ or χ tend to zero independently. If λ and χ are both greater than unity, there is a boundary layer in which the variation of temperature is exponential and which tends to a discontinuity as $\lambda \to \infty$. If $\chi \gg \lambda^2$, $\beta/\bar{\beta}$ becomes a function of λ only; figure 1 shows a number of profiles for this limiting case.

5. Approximate solutions

Pellew & Southwell showed that variational methods can lead to remarkably accurate values of the critical Rayleigh number even when the precise form of w is not known. The basis of the method is as follows. Let the right hand side of either (27) or (28) be written $Q(w)$ where Q is one of two operators both of which, with the boundary conditions (11), can be shown to be Hermitian. Consider the function

$$R' = \frac{\int_{-\frac{1}{2}}^{+\frac{1}{2}} w'Q(w')\, d\zeta}{\int_{-\frac{1}{2}}^{+\frac{1}{2}} w'Q(w'^2)\, d\zeta}, \tag{41}$$

where w' is any function which satisfies the boundary conditions (11). It can be shown that R' is a minimum when w' is the solution of (27) or (28) which leads to the lowest possible Rayleigh number, i.e. the critical Rayleigh

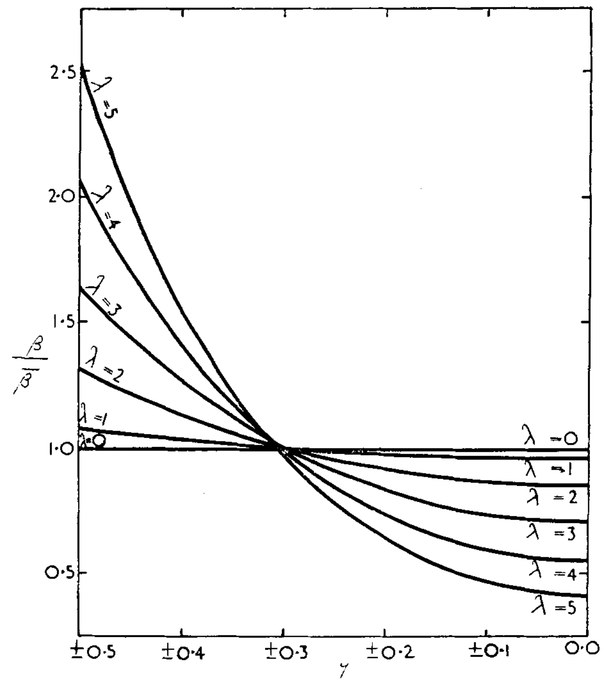

Figure 1. β/β as a function of ζ for the limiting case $\chi \gg \lambda^2$.

number R_c. Thus, because of (27) or (28), the minimum of R' will be equal to R_c.

The procedure is to choose a physically reasonable function w' which satisfies the boundary conditions and contains one variable parameter. R' is evaluated and minimized with respect to this variable parameter.

Two functions have been investigated representing two general types of motion which may reasonably be expected to occur. The first,

$$w' = \sin n\pi(\zeta + \tfrac{1}{2}), \tag{42}$$

is the correct symmetrical solution of the conventional Rayleigh problem. In this case the variable parameter is n, which can, however, take only integral values. This function was chosen to represent a disturbance taking place through the whole body of the fluid.

Using the function (42), the integrals in (41) are easily evaluated for both the 'transparent' and 'opaque' approximations. The resulting expressions contain both a^2 and n. The value of a^2 which makes R' a minimum determines the width of the cells in the least stable mode of motion. For the 'opaque' approximation the minimum lies at

$$a^2 = \frac{n^2\pi^2}{2}, \tag{43}$$

which is the same result as that obtained in the conventional Rayleigh problem. For the 'transparent approximation' the minimum of R' lies at a value of a^2 lying between (43) and

$$a^2 = n^2\pi^3. \tag{44}$$

In this approximation, therefore, the cell width is variable, but whenever radiative effects are large one finds that (44) is valid.

When R' has been minimized with respect to a^2, it is clear by inspection that the smallest value R' occurs for $n = 1$. The same result is, of course, found in the conventional Rayleigh problem.

Approximate values of R_c computed in this way are shown in figure 2 for $\chi = 10^3$, 10^6 and 10^9, and for λ between 10^{-1} and 10^7. The 'transparent' and 'opaque' approximations are shown in their respective ranges of validity. Discussion of these results will be taken up in the next section.

The second form chosen for w' is

$$w' = (A+B\zeta^2)\coth\eta\zeta + (C\zeta+\zeta^3)\sinh\eta\zeta, \tag{45}$$

where A, B, and C are determined from the boundary conditions (11) and η is the variable parameter. If $\eta \gg 1$, this expression is very large for values of ζ within about η^{-1} of the boundary values. The reason for adopting this function is that the form of $\beta/\bar{\beta}$ given by (39) suggests that motions in the temperature boundary layer might possibly become unstable before motions through the body of the fluid, and therefore that a function representing motions concentrated in a boundary layer should be tried.

It was conjectured that, for small η, the critical Rayleigh numbers derived from (45) would not differ greatly from those derived from (42), since in this case both functions represent motions though the body of the fluid. Since we are mainly concerned with results differing greatly from (42), it was considered to be sufficient to use (45) in its asymptotic form for large η and to neglect contributions to the integrals in (41) from the neighbourhood of $\zeta = 0$. This greatly reduces the labour of the computations, and it appears to be fully justified by the way in which, in figure 2, the full lines and chain dashed and dotted lines converge as λ decreases.

In the half-space $\zeta > 0$, the asymptotic form of (45) which fulfills the boundary conditions (11) is

$$w' = \tfrac{1}{2}e^{\eta\zeta}\left[\left(\zeta - \tfrac{1}{2} - \frac{1}{\eta}\right)^3 + \frac{1}{\eta^3}\right]. \tag{46}$$

Equation (46) and the appropriate operators for the 'opaque' and 'transparent' approximations can now be substituted in (41), and expressions for R' can be found in terms of a^2 and η. In doing so, terms of the order of χ^{-1} were neglected in comparison with unity, since the minimum value of χ used in the computations is 10^3.

Figure 2. Approximate critical Rayleigh numbers as a function of ζ for $\chi = 10^3, 10^6, 10^9$. The full lines, marked 'body motions', are computed from the function (42) for both the 'opaque' and 'transparent' approximations. The dashed line is an attempted interpolation over the range where neither approximation is valid. The chain dashed and dotted line marked 'boundary layer motions' comes from the use of the function (45) with the 'transparent' approximation.

In the case of the 'opaque' approximation, it turns out that R' has a minimum with respect to a^2 but not with respect to η. This suggests that the 'opaque' approximation is never valid for these boundary layer motions.

In the case of the 'transparent' approximation, the minimum value of R' was found to lie at $\lambda^2/\eta^2 = 5{\cdot}11$, $a^2/\eta^2 = 0{\cdot}372$, giving as an estimate of the critical Rayleigh number

$$R_c = 11{\cdot}89\frac{\lambda^4}{\chi}\left[\frac{2\chi}{\lambda} + \frac{\surd(3+3\chi)}{2} + 1\right]. \tag{47}$$

Computations from (47) are shown in figure 2 for $\lambda > 10$, which implies $\eta > 10(5\cdot 11)^{-1/2} = 4\cdot 4$, a value large enough for (46) to be a good approximation to (45).

The 'transparent' approximation is valid provided that

$$\kappa^2 h^2/a^2 = \lambda^2/\{3a^2(1+\chi)\} \ll 1.$$

Since the minimum of R' lies at $\lambda^2 = 13\cdot 7a^2$, this inequality is always satisfied provided $\chi \gg 4$, so that the transparent approximation provides a complete solution for large χ. Physically this means that, since the radiation boundary layer is always smaller than the radiation mean-free-path (boundary layer thickness $\doteqdot h\lambda^{-1} = \kappa^{-1}[3+3\chi]^{-1/2} \ll \kappa^{-1}$ for $\chi \gg 1$), motions occurring in it can always be treated by the 'transparent' approximation.

Discussion

Let us first consider the 'body motion' curves in figure 2 obtained by the use of the function (42). The most remarkable result is that there is almost no doubt about the form of the solution in the region where neither approximate treatment is valid. This means that a reasonably complete and satisfactory solution to the problem has been achieved without entering into the usual intricacies associated with radiative transfer problems. There is therefore every reason to hope that similar methods used on more realistic models will yield results of real importance to the study of planetary and stellar atmospheres.

As far as this particular problem is concerned, the results indicate that if either λ or χ is independently less than unity, then there are no radiative effects on the convective motions; they also indicate that $R_c = 657$, as in the conventional Rayleigh problem. If λ and χ are both greater than unity, then the fluid is stabilized. This stabilization has two aspects. First, the effect of radiation on the initial static state is to concentrate the temperature variations into boundary layers, leaving the body of the fluid in a more stable state. Secondly, the radiative transfer tends to damp out any motions which may arise by providing a means of heat transfer from hotter to colder parts of the fluid in addition to the molecular diffusion.

The maximum stabilization is achieved in the case of body motions if the fluid is opaque, and then the critical Rayleigh number is $657(1+\chi)$. This is exactly the result that would be obtained from the conventional Rayleigh problem with a diffusivity $k(1+\chi)$ in place of k. The same result was obtained by Brunt (1944).

With regard to the 'boundary layer' curves in figure 1, the most remarkable fact is again that an approximate solution has been obtained by comparatively simple methods, and in this case the solution is valid for all values of λ under consideration. For large values of λ, when the 'opaque' approximation is valid for the body motions, the critical Rayleigh numbers are much higher in the case of the boundary layer motions, and so these motions should not occur. For smaller values of λ, the two sets of curves converge, and in general the boundary layer motions are more stable except near $\lambda = 10^2$ for $\chi = 10^9$. It is very doubtful whether much physical significance should be attached to the apparent change of regime that the mathematical analysis indicates. It is much more likely that, where the boundary layer and body motion curves lie close to each other, the motion is very much more complicated than equations (42) and (45) would suggest, and that they both give approximate critical Rayleigh numbers which are considerably too high.

The results presented here may have some bearing on a suitable laboratory experiment. Large radiative effects are more likely if a gas rather than a liquid is used as a fluid; but, unfortunately, radiative transfer in a gas at normal temperatures will be by vibration–rotation bands of great complexity for which a mean absorption coefficient has little meaning. Nevertheless, a crude estimate can be made for water vapour at S.T.P. from data given by Cowling (1950); this is $\kappa = 2 \times 10^{-2}\,\text{cm}^{-1}$ (the uncertainty in this figure may be as high as an order of magnitude). With $\pi \underset{\sim}{Q} = 6 \times 10^3\,\text{erg cm}^{-2}\,\text{sec}^{-1}\,\text{deg}^{-1}$ (corresponding to 273°K) and ks (thermal conductivity) $= 2 \times 10^3\,\text{erg cm}^{-1}\,\text{sec}^{-1}\,\text{deg}^{-1}$, there results $\chi = 2 \times 10^2$ and, if $h = 10\,\text{cm}$, $\lambda = 3$. Inspection of figure 2 suggests that the critical Rayleigh number might be twice as great as without radiative transfer, which is a difference which should be detected easily in the laboratory.

Planetary and stellar atmospheres have been mentioned, and it is of interest to record the magnitudes involved. In the earth's atmosphere χ might be of the order of 10^5, and motions on the scale of 1 km can be treated with the 'opaque' approximation. Just beneath the surface of the solar photosphere (van der Hulst 1953, Minnaert 1953), χ is of the order of 10^{12}, and motions on the same scale as a solar granule must be treated with the 'opaque' approximation.

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